Geometrical measures for (mildly) non-Gaussian cosmological fields

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with: S. Codis, C. Gay, T. Sousbie, C. Pichon (IAP)

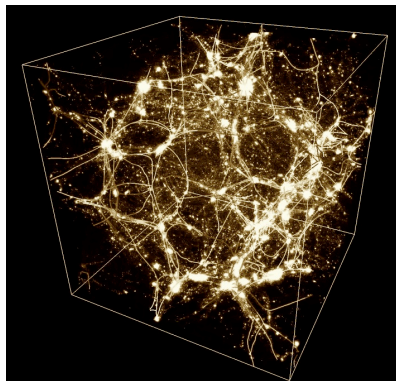# Geometry of random fields: 3D

Cosmology

Doroshkevich, 1970
Arnold, Zeldovich, Shandarin, 1982
Bond, Bardeen, Kaiser, Szalay, 1986

Geometrical questions:

- abundance and shapes of peaks of different scale
- Length of filamentary bridges
- Connectivity of peaks by filamentary ridges
- Relation between the properties of filaments and clusters
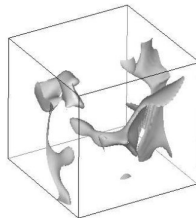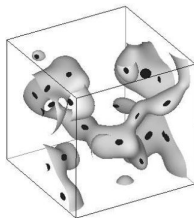

Sims: Horizon project + skeleton

From Peaks to Filaments, Skeleton of the Cosmic Web

- The high rare peaks of the field largely define how large scale structure looks like.

- The Skeleton of LSS traces the filamentary bridges that form the Web between them and provides the next level of detail to LSS understanding.
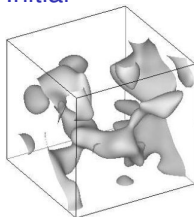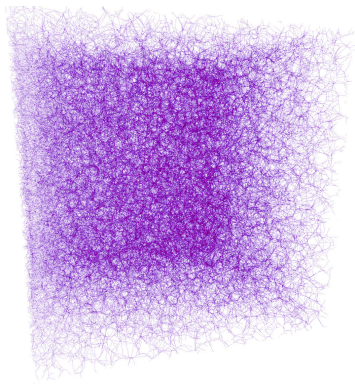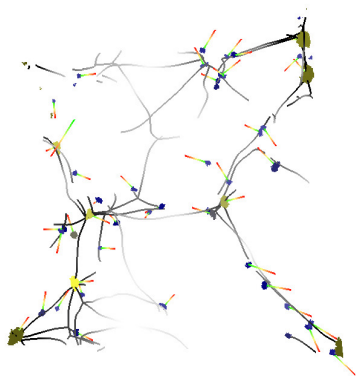
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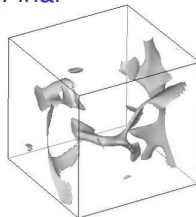
Skeleton as tracer of the filaments

Dark matter skeleton in 4π simulation

Halo spin vs the skeleton
Codis et al, 2012

Geometrical measures for random fields

Starting point

Let us think about such properties of a random field ρ as Euler characteristic (genus), density of maxima, length of skeleton. Their computation reguire knowledge of the joint distribution

$$\mathscr{P}(\rho, \rho_i, \rho_{ij}, \ldots)$$

of the field ρ and its first ρ_i, second ρ_{ij} (Hessian matrix) and perhaps higher derivatives, for instance

$$n_{max}(v) = \int_{0 \geq \lambda_1 \geq \lambda_2 \geq \ldots} \mathscr{P}(\rho = v, \rho_i, \rho_{ij}) \delta(\rho_i) |\rho_{ij}| d\rho_{ij}$$

Usual approach is to deal with it in the Hessian eigenvalue space, since that's where the boundary conditions are the simplest.

Towards the local description of the Skeleton
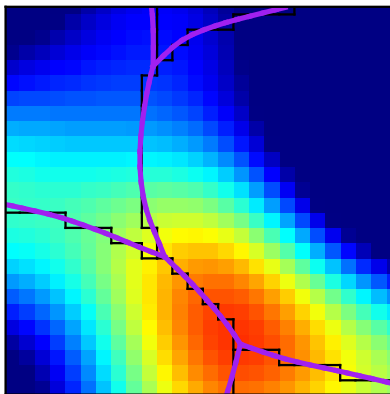
Towards the local description of the Skeleton

$$(\nabla\nabla\rho)\cdot\nabla\rho = \lambda\nabla\rho$$
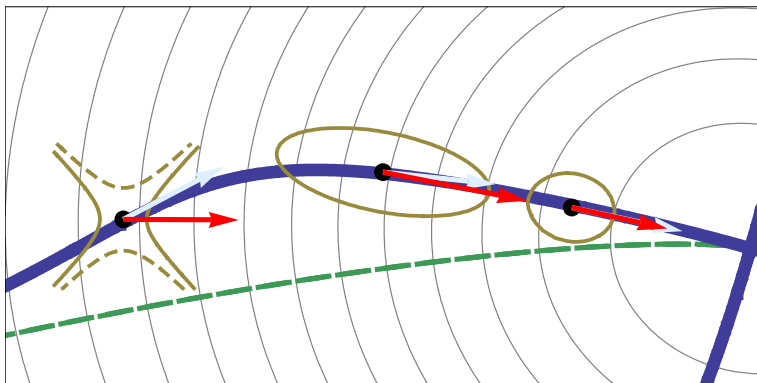
Local description of the Skeleton

Definition

Along the ridge the gradient of the field $\nabla\rho$ is aligned with the direction of the least curvature. Formally, skeleton is the set of points where (with additional restrictions)

$$S \equiv (\nabla\nabla\rho \cdot \nabla\rho) \times \nabla\rho = 0$$

Features

Local definition introduces point process that allows analytical treatment similar to statistics of extrema.

Stiff Approximation: Theory fits Simulations (of course)

$$\left(\frac{\partial \mathscr{L}}{\partial v}\right)_{\text{stiff}} \propto \frac{1}{R_*^{N-1}} \int d\lambda_1 \ldots d\lambda_N \mathscr{P}(v, \lambda_1, \ldots, \lambda_N) |\lambda_2 \cdots \lambda_N|$$
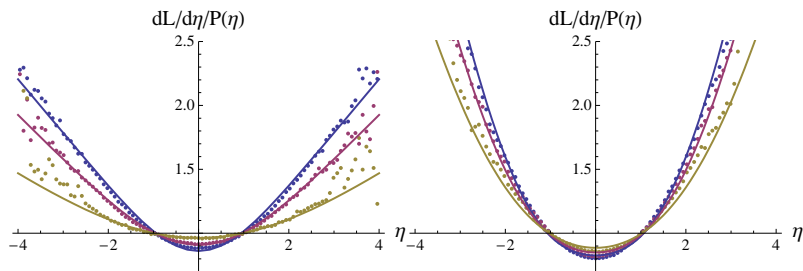
Notable Result:

At high densities $v = \rho/\sigma_0$ density of filaments is enhanced

$$2D: \propto \gamma v \ , \ \ 3D: \propto (\gamma v)^2$$

Non-Gaussian expansion for geometrical statistics

- To treat non-Gaussianities the idea is to expand $\mathscr{P}(\rho, \rho_i, \rho_{ij})$ into orthogonal polinomials around the Gaussian approximation like
$$P(x) = G(x)\left(1 + \sum_n \langle x^n \rangle_c H_n(x)\right)$$

- The trick to avoid difficulties is an appropriate choice of variables:
  - that are invariant wrt symmetries of the problem (isotropy)
  - that are polynomial in the field quanities (λ's are no good)
  - that simplify the Gaussian limit, being as uncorrelated as possible

- Useful set is: $I_1, \cdots, I_N, \ q^2, \ \zeta \equiv \frac{\rho + \gamma I_1}{1 - \gamma^2}$
  where I_n are N polynomial rotation invariants of the Hessian matrix ρ_{ij},
  $$I_1 = \mathrm{Tr}\,\rho_{ij} \quad , \dots , \quad I_N = \det|\rho_{ij}|$$
  (and $I_2 \dots I_{N-1}$ are built from the minors of orders 2 to N-1)

- Actually, better to use more 'irreducible' combinations J_i, in ND-space

$$J_1 = I_1 \ , \quad J_{s \geq 2} = I_1^s - \sum_{p=2}^s \frac{(-N)^p C_s^p}{(s-1) C_N^p} I_1^{s-p} I_p$$

Orhogonal polymonial expansion for 2D $\mathscr{P}(\rho, q^2, J_1, J_2)$

Gaussian limit JPDF

$$G_{2D}(\zeta, q^2, J_1, J_2)\, \mathrm{d}\zeta \mathrm{d}q^2 \mathrm{d}J_1 \mathrm{d}J_2 = \frac{1}{2\pi} e^{-\frac{1}{2}(\zeta^2 + 2q^2 + J_1^2 + 2J_2)}\mathrm{d}\zeta \mathrm{d}q^2 \mathrm{d}J_1 \mathrm{d}J_2$$

serves as the weight for defining the expansion polynomials in

- ζ, J_1 – ($[-\infty, \infty]$, gaussian weight) – Hermite
- q^2, J_2 – ($[0, \infty]$, exponential weight) – Laguerre.

$$\mathscr{P}_{2D}(\zeta, q^2, J_1, J_2) = G_{2D} \times \Big[1 +$$

$$\sum_{n=3}^{\infty} \sum_{i,j,k,l=0}^{i+2j+k+2l=n} \frac{(-1)^{j+l}}{i!\, j!\, k!\, l!} \left\langle \zeta^i q^{2j} J_1^{\ k} J_2^{\ l} \right\rangle_{\mathrm{GC}} H_i(\zeta) L_j\left(q^2\right) H_k(J_1) L_l(J_2) \Big]$$

This is an expansion to all orders in powers of the field n.

"Gram-Charlier" coefficients of expansion $\langle \ldots \rangle_{\mathrm{GC}}$

are

$$
\begin{aligned}
\left\langle \zeta^i q^{2j} J_1^k J_2^l \right\rangle_{\mathrm{GC}} &= \frac{j!\, l!}{(-1)^{j+l}} \left\langle H_i(\zeta) L_j\left(q^2\right) H_k(J_1) L_l(J_2) \right\rangle \\
&= \left\langle \zeta^i q^{2j} J_1^k J_2^l \right\rangle + \ldots
\end{aligned}
$$

which for order $n = i + k + 2j + 2l$

$n = 3$ are equal to the simple moments, $\langle \zeta J_2 \rangle_{\mathrm{GC}} = \langle \zeta J_2 \rangle$

$n \le 5$ are equal to cumulants, $\langle \zeta q^2 J_1 \rangle_{\mathrm{GC}} = \langle \zeta q^2 J_1 \rangle_c$

     where in the cumulants our quadratic variables are
     reverted to field quantities

$\Rightarrow$ expansion coefficients can be predicted by pertrubation theories

Euler characteristic (genus) as a function of threshold

General expression in ND

$$\frac{\chi(v)}{2} = (-1)^N \int_v^\infty dx \int dq^2 q^{N-1} \delta_D^N(q^2) \int \prod_{s=1}^N dJ_s \mathscr{P}_{\mathrm{ND}}(\ldots) I_N$$

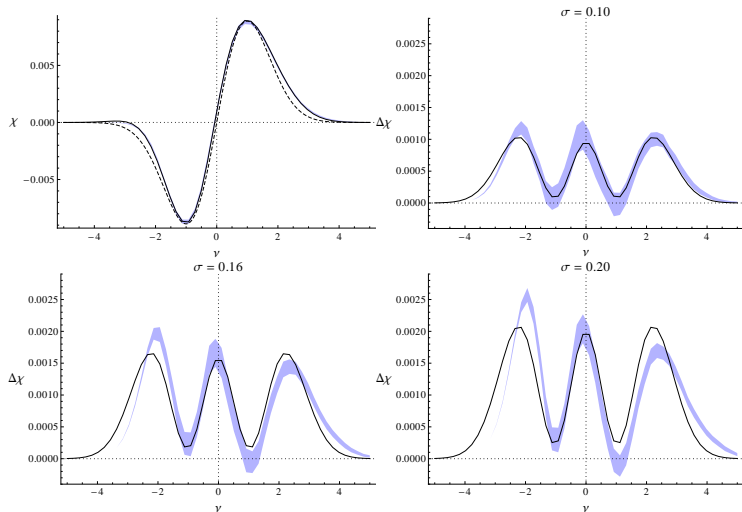can be integrated to give "moment" expansion to all orders

$$\chi(v) = \frac{1}{\sqrt{2\pi}R_*} \exp\left(-\frac{v^2}{2}\right) \times \frac{2}{(2\pi)^{N/2}} \left(\frac{\gamma}{\sqrt{N}}\right)^N \left[H_{N-1}(v) + \right.$$

$$\left. + \sum_{n=3}^\infty \sum_{s=0}^N \gamma^{-s} \sum_{i,j=0}^{i+2j=n-s} \frac{(-N)^{j+s}(N-2)!!L_j^{(\frac{N-2}{2})}(0)}{i!(2j+N-2)!!} \left\langle x^i q^{2j} I_s \right\rangle_{GC} H_{i+N-s-1}(v) \right]$$

(cf first term: Matsubara 1994-2005)

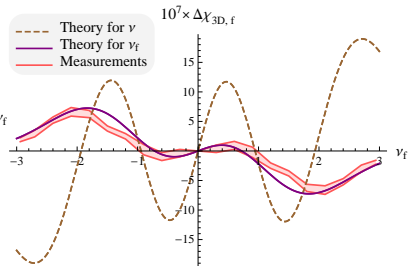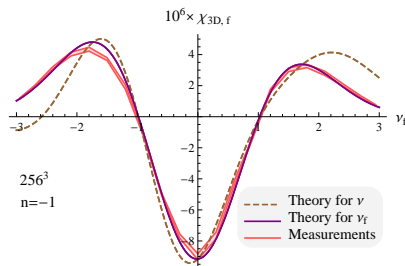with coefficients that can be predicted by perturbation theories

How non-Gaussianity develops, eq 2D Euler charachteristic

Excitation of Hermite modes of alternating parity

Statistics versus observable filling factor

ν_f is the filling factor expressed as threshold units via its non-Gaussian expansion. $\sigma = 0.18$ (with parity cleaning)
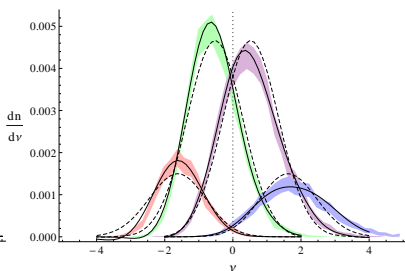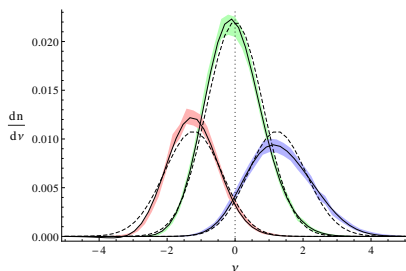
Extrema counts and more

Formalism to study popular and novel geometrical statistics in non-gaussian regime

- Genus in ND is fully described
- 2D Extrema density (CMB case) computed analytically
- 3D Extrema density (LSS)

$$
n_{\mp--} = \frac{29\sqrt{15}\mp18\sqrt{10}}{1800\pi^2 R_*^3} + \frac{5\sqrt{5}}{24\pi^2\sqrt{6\pi}R_*^3}\left(\left\langle q^2 J_1\right\rangle - \frac{8}{21}\left\langle J_1^3\right\rangle + \frac{10}{21}\left\langle J_1 J_2\right\rangle\right)
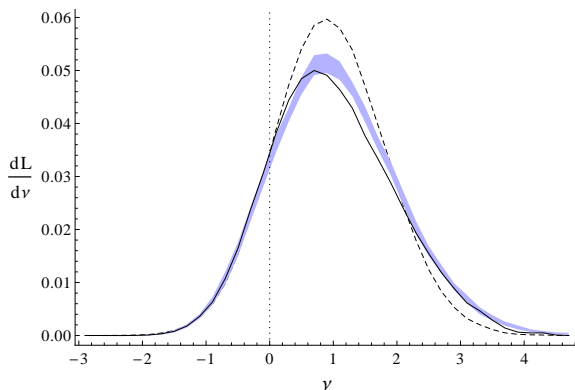$$

$$
n_{++\pm} = \frac{29\sqrt{15}\mp18\sqrt{10}}{1800\pi^2 R_*^3} - \frac{5\sqrt{5}}{24\pi^2\sqrt{6\pi}R_*^3}\left(\left\langle q^2 J_1\right\rangle - \frac{8}{21}\left\langle J_1^3\right\rangle + \frac{10}{21}\left\langle J_1 J_2\right\rangle\right)
$$

- Peak shapes, skeleton length, correlation of velocity flow and filaments

Non-Gaussian Skeleton length (Gay et al. 2012)

Extension to anisotropic statistics

- In cosmology we do not always deal with isotropic fields, or statistics

- Obvious example - density in the redshift space

- more subtle case – skeleton statistics of the filaments are not isotropic !
  (there is a angle dependence between local directions of gradient and
  Hessian)

- The generalization of our approach is straightforward - use symmetries that
  remain, define polynomial variables, find uncorrelated combinations and
  expansion polynomials as determined by the Gaussian limit.

- Done for the skeleton in Gay, Pichon, Pogosyan, 2012 and for the redshift
  space in Codis, Pichon, Pogosyan, Bernardeau, Matsubara 2013. for the
  redshift space

Polymonial expansion for $\mathscr{P}_{3D}$ in redshift space

Symmetry:

Rotational around the line of sight (LOS along 3rd coordinate)

Variables:

$$
\begin{aligned}
\text{linear(4)} \quad & x,\, x_3,\, x_{33},\, J_{1\perp} = x_{11} + x_{22} \\
\text{quadratic(3)} \quad & q^2 = x_1^2 + x_2^2,\, Q^2 = x_{13}^2 + x_{23}^2,\, J_{2\perp} = (x_{11} - x_{22})^2 + 4x_{12}^2 \\
\text{cubic(1)} \quad & \Upsilon = \left(x_{13}^2 - x_{23}^2\right)(x_{11} - x_{22}) + 4x_{12}x_{13}x_{23}
\end{aligned}
$$

Gaussian limit JPDF

$$
G(x, q_\perp^2, x_3\,\zeta, J_{2\perp}, \xi, Q^2, \Upsilon) = \frac{1}{4\pi^3 \sqrt{Q^4 J_{2\perp} - \Upsilon^2}} e^{-\frac{1}{2}x^2 - q_\perp^2 - \frac{1}{2}x_3^2 - \frac{1}{2}\zeta^2 - J_{2\perp} - \frac{1}{2}\xi^2 - Q^2}
$$

Uniformly distributed $\Upsilon \in \left[-Q^2\sqrt{J_{2\perp}}, +Q^2\sqrt{J_{2\perp}}\right]$ can be integrated over for Minkowski functionals, extrema ...
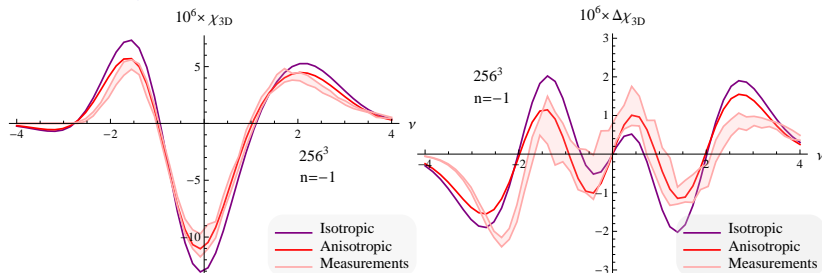
Euler characteristic in redshift space

$$\chi_{2+1}(v) = \frac{e^{-v^2/2}}{8\pi^2} \left[\frac{\sigma_{1\parallel}\sigma_{1\perp}^2}{\sigma^3} H_2(v) + \sum_{n=3}^{\infty} \chi_{2+1}^{(n)} \right]$$

with non-Gaussian corrections $\chi_{2+1}^{(n)}$, given, to all orders, by

$$
\begin{aligned}
\chi_{2+1}^{(n)}(v) &= \frac{\sigma_{2\perp}^2 \sigma_{2\parallel}}{\sigma_{1\perp}^2 \sigma_{1\parallel}} \left[\sum_{\sigma_n} \frac{(-1)^{j+m}}{2^m i! \, j! \, m!} H_{i+2}(v) \gamma_\parallel \gamma_\perp^2 \left\langle x^i q_\perp^{2j} x_3^{2m} \right\rangle_{\text{GC}} \right. \\
&\quad - \sum_{\sigma_{n-1}} \frac{(-1)^{j+m}}{2^m i! \, j! \, m!} H_{i+1}(v) \left\langle x^i q_\perp^{2j} x_3^{2m} \left(\gamma_\perp^2 x_{33} + 2\gamma_\perp \gamma_\parallel J_{1\perp} \right) \right\rangle_{\text{GC}} \\
&\quad + \sum_{\sigma_{n-2}} \frac{(-1)^{j+m}}{2^m i! \, j! \, m!} H_i(v) \left\langle x^i q_\perp^{2j} x_3^{2m} \left(2\gamma_\perp (J_{1\perp} x_{33} - \gamma_2 Q^2) + \gamma_\parallel (J_{1\perp}^2 - J_{2\perp}) \right) \right\rangle_{\text{GC}} \\
&\quad \left. - \sum_{\sigma_{n-3}} \frac{(-1)^{j+m}}{2^m i! \, j! \, m!} H_{i-1}(v) \left\langle x^i q_\perp^{2j} x_3^{2m} \left(x_{33}(J_{1\perp}^2 - J_{2\perp}) - 2\gamma_2 \left(Q^2 J_{1\perp} - \Upsilon \right) \right) \right\rangle_{\text{GC}} \right]
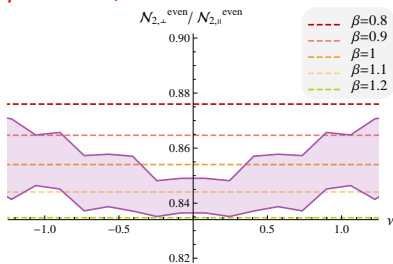\end{aligned}
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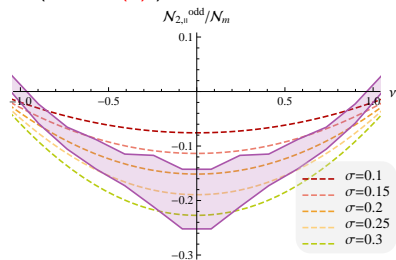# Euler characteristics in redshift space

$\sigma = 0.18,\ f = 1$

Using non-Gaussianity and redshift distortions of geometrical measures of the Cosmic Web to recover

$\beta = \Omega^{0.55}/b$

Reconstructed β from angular dependence of Minkowski functionals in 2D slices

σ (and D(z))

Reconstructed σ from non-Gaussian corrections of Minkowski functionals

Conclusions

- We progressed beyond fixed-order non-gaussian corrections, to complete representation of geometrical statistics via the moments that are predicted by the theory. Examples are perturbation theory of the gravitational clustering or non-linear effects at inflationary stage.

- Direct utilization of the symmetries of the problem makes many previously untractable calculations possible.

- We now know the non-gaussian expansion to all orders (and in arbitrary dimension) for the Euler statistic, and isosurface area.

- Dependence of extrema count on non-gaussian corrections is understood.

- and so is of the differential length of the (local) skeleton.

- Extension of the formalism to redshift space has been developed.

- The path is open for application of geometrical descriptors to CMB data, $D(z)$ reconstruction, etc.